Effects of fertilizers (Control, Nitro, Super, Super + Nitro) on grain yield are analyzed. Our treatments split into combinations of two "**factors**" with two "**levels**" each:

Factor 1		Factor 2
0 Super	×	0 Nitro
Super		Nitro

		Nitro 0	Nitro 1
Super	0	μ_1	μ_2
	1	μ_3	μ_4

This study design is a **factorial design**. Our statistical analysis technique is called **factorial ANOVA** or **two-way ANOVA**.

If Super and Nitro do not interact, the effect of Super + Nitro is

$$(\text{Nitro Effect}) + (\text{Super Effect}) = (\mu_2 - \mu_1) + (\mu_3 - \mu_1)$$

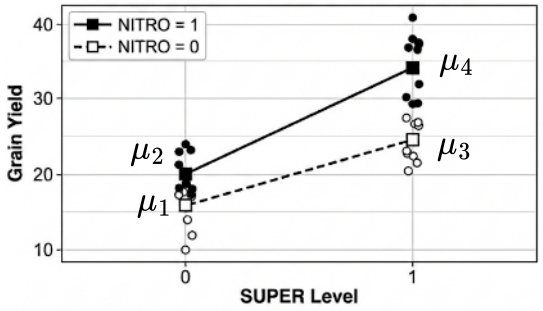
The actual observed effect is:

$$(\text{Effect of Super + Nitro}) = \mu_4 - \mu_1$$

The difference between these measures the **interaction** between Super vs Nitro:

$$(\mu_4 - \mu_1) - [(\mu_2 - \mu_1) + (\mu_3 - \mu_1)] = \mu_1 - \mu_2 - \mu_3 + \mu_4 \quad \Longrightarrow \quad c = (1, -1, -1, 1)$$

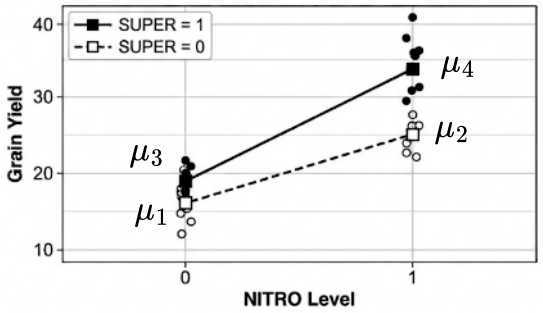
Two more ways to interpret this contrast:



(Black Line Slope) - (Dotted Line Slope)

$= (\mu_4 - \mu_2) - (\mu_3 - \mu_1) = \mu_1 - \mu_2 - \mu_3 + \mu_4$

$\Rightarrow c = (1, -1, -1, 1)$



(Black Line Slope) - (Dotted Line Slope)

$= (\mu_4 - \mu_3) - (\mu_2 - \mu_1) = \mu_1 - \mu_2 - \mu_3 + \mu_4$

$\Rightarrow c = (1, -1, -1, 1)$

```
data grain_yields;
 input super nitro y;
 datalines;
0 0 19.2
0 0 18.4
0 0 17.0
0 0 17.6
0 0 17.2
1 0 18.2
1 0 19.8
1 0 19.4
1 0 19.0
1 0 19.8
0 1 20.0
0 1 21.6
0 1 22.0
0 1 20.8
0 1 20.4
1 1 23.6
1 1 21.6
1 1 23.2
1 1 21.4
1 1 21.2
;
run;
```

μ_1 μ_3 μ_2 μ_4

```
proc glm data=grain_yields;
  class super nitro;
  model y = super nitro super*nitro;
  contrast 'Main Effect: Super' super -1 1;
  contrast 'Main Effect: Nitro' nitro -1 1;
  contrast 'Interaction: Super * Nitro' super*nitro 1 -1 -1 1;
run;
```

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	3	54.07000000	18.02333333	22.47	<.0001
Error	16	12.83200000	0.80200000		
Corrected Total	19	66.90200000			

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
Main Effect: Super	1	8.45000000	8.45000000	10.54	0.0051
Main Effect: Nitro	1	45.60200000	45.60200000	56.86	<.0001
Interaction: Super * Nitro	1	0.01800000	0.01800000	0.02	0.8828

If p-value is significant, there is statistically significant difference in avg response with Super vs without Super.

If p-value is significant, avg response with Nitro vs without Nitro is proven to differ.

If p-value is significant, there is statistically significant interaction between effect of Nitro vs Super on avg grain yield.

To get a complete set of orthogonal contrasts for 2-way ANOVA, compute:

- Main effect contrasts: use contrasts for factor A & factor B, resp.
- Interaction contrasts: Kronecker products of the factor A vs B contrasts.

Ex.

$$c_1 \otimes d_1 = \begin{pmatrix} -2 \cdot d_1 \\ 1 \cdot d_1 \\ 1 \cdot d_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \\ -1 \\ -1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

		Factor B (crop)				c_1	c_2
		Legumes		Nonlegumes		A contrasts	
		Lupins	Peas	Mustard	Barley		
Factor A (moisture)	Dry	2	2	-2	-2	-2	0
	25% asm	-1	-1	1	1	1	-1
	50% asm	-1	-1	1	1	1	1
B contrasts	d_1	-1	-1	1	1		
	d_2	-1	1	0	0		
	d_3	0	0	-1	1		

Theory: Two-way ANOVA model decomposes the spaces into the orthogonal basis:

$$\begin{bmatrix} \mu_{A1} \\ \mu_{A2} \\ \mu_{A3} \end{bmatrix} \in \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, c_1, c_2 \right)$$

$$\begin{bmatrix} \mu_{B1} \\ \mu_{B2} \\ \mu_{B3} \\ \mu_{B4} \end{bmatrix} \in \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, d_1, d_2, d_3 \right)$$

$$\rightarrow \quad \begin{bmatrix} \mu_{A1} \\ \mu_{A2} \\ \mu_{A3} \end{bmatrix} \otimes \begin{bmatrix} \mu_{B1} \\ \mu_{B2} \\ \mu_{B3} \\ \mu_{B4} \end{bmatrix} \in \text{span} \begin{bmatrix} \text{(grand mean vector)} \\ \vec{1}_3 \otimes \vec{1}_4 = \vec{1}_{12} \\ \text{(factor A contrasts)} \\ c_i \otimes \vec{1}_4 \\ \text{(factor B contrasts)} \\ \vec{1}_3 \otimes d_j \\ \text{(interaction effects)} \\ c_i \otimes d_j \end{bmatrix}$$